Quantum mechanics II, Chapter 1 : Basics

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Even space bears can find themselves in superposition.

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Problem 1 : Bloch sphere for pure states

This problem is *optional.* If you have done similar problems before feel free to skip. But the Bloch sphere is important so do only skip if you're already comfortable with this material.

1. Show that any quantum state $|\psi\rangle$ of a 2 level system with classical states $|0\rangle$ and $|1\rangle$ can be written
$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle\,, \quad \theta \in [0,\pi], \quad \phi \in [0, 2\pi], \tag{1}$$
and conclude that the unit sphere in 3D allows for the representation of a pure state of a 2 level system (qubit, spin 1/2, photon,...). We call this unit sphere the Bloch sphere
2. In the case of a 1/2 spin, the convention for the north pole is $|0\rangle$ (resp. the south pole $|1\rangle$) which is eigenstate of $S_z = \frac{\hbar}{2}\sigma_z$ with eigenvalue $+\hbar/2$ (resp. $-\hbar/2$). Show that state $|\psi\rangle$ of Eq. (1) is the eigenstate of $\boldsymbol{\sigma_n} = \boldsymbol{\sigma}.\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector with direction parametrized by θ and ϕ in spherical coordinates and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. In a classical picture of the spin, $\boldsymbol{n}$ would then designate the direction in which that spin is pointing.
3. Sketch on the Bloch sphere the effect of applying the operators σ_x, σ_y and σ_z to an arbitrary state $|\psi\rangle$.
4. Prove that the Pauli matrices together with the identity form an orthogonal basis (under inner-product $\langle A, B\rangle = \mathrm{Tr}\big[A^\dagger\, B\big]$) of all operators acting on a 2 level system.
5. Show that
$$e^{-i\vartheta\mathbf{n}.\boldsymbol{\sigma}} = \cos(\vartheta)\mathbb{I} - i\sin(\vartheta)\mathbf{n}.\boldsymbol{\sigma}\,. \tag{2}$$
6. Hence sketch the action of $e^{-i\vartheta\mathbf{n}.\boldsymbol{\sigma}}$ of an arbitrary state on the Bloch sphere.
7. Compute the expectation value of the Pauli operators σ_x, σ_y and σ_z for the arbitrary state $|\psi\rangle$. Explain your answer in terms of the Bloch sphere.

Problem 2 : Rethinking the thought experiment

Consider the thought experiment described in Section 1.4 of the course notes. I'll copy it over here for simplicity.

Thought experiment 1 : We start with a system in state $|0\rangle$. We wait half an hour (in our units, from $t = 0$ to $t = 1/2$) before measuring it. We then find that 50% of the time it is in state $|1\rangle$ and that 50% of the time it is in state $|0\rangle$.

Thought experiment 2 : We start with a system in state $|1\rangle$. We wait half an hour before measuring it. We then find that 50% of the time it is in state $|1\rangle$ and that 50% of the time it is in state $|0\rangle$.

Let $U = e^{-iHt}$ represent the evolution of the system as we "wait for t hours". The state of the system pre-measurement in the two experiments can therefore be written as $|\psi\rangle = U|0\rangle$ and $|\phi\rangle = U|1\rangle$ respectively.

1. Write down the most general form for the states $|\psi\rangle$ and $|\phi\rangle$ that are consistent with the observed outcomes in experiments 1 and 2.

Thought experiment 3 : We start with a system in state $|0\rangle$. We wait half an hour before measuring it. We then find that 50% of the time it is in state $|1\rangle$ and that 50% of the time it is in state $|0\rangle$. Then we wait another half an hour before measuring again. We then find that 50% of the time it is in state $|1\rangle$ and that 50% of the time it is in state $|0\rangle$.

2. Use your answer to (a) to explain the result of the third thought experiment.

Thought experiment 4 : We start with a system in state $|0\rangle$. We *wait a full hour* before measuring it. We find that the system is always in state $|0\rangle$.

For concreteness, let's now assume that $|\psi\rangle = U|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

3. Write down two non-equivalent [1] unitaries that U could be and the corresponding Hamiltonian H (hint - think about Problem 1) which would generate them. (Are there more than two possibilities here ?)
4. Compute the corresponding state $|\phi\rangle$ for your two cases.
5. Given that in experiment 4 the system was always found in $|0\rangle$, what can we say about the Hamiltonian of the system ?

1. i.e. not the same up to a global phase

Problem 3 : Collection of tensor product exercises

Consider a state $|\psi\rangle$ of *five* quantum particles, each with two levels denoted $|0\rangle$ and $|1\rangle$, which is initially

$$\begin{aligned} |\psi\rangle &= |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \\ &= |00000\rangle . \end{aligned}$$

We will identify these five particles with an index $0 \leq i < 5$ beginning from zero, which indicates the rightmost *ket* above.

1. Write down the state produced by applying the σ_x operator to the rightmost particle (of index $i = 0$). Write the *bra* version of it.
2. Write down the state produced by applying the σ_y operator to particle $i = 3$. Write the *bra* version of it.
3. For two arbitrary matrices A, B compute the commutator of $[A \otimes 1, 1 \otimes B]$ and $\{A \otimes 1, 1 \otimes B\}$. Recall that $[A, B] = AB - BA$, $\{A, B\} = AB + BA$.
4. The exponential of a sum of matrices satisfies $e^{A+B} = e^A e^B$ if $[A, B] = 0$. Use this and the previous exercise to show that $e^{A\otimes 1+1\otimes B} = e^A \otimes e^B$
5. Write the matrix version of the one-qubit operator $a\,|0\rangle\langle 1| + b\,|1\rangle\langle 1|$, where a and b are arbitrary scalars.
6. Let A, B be two arbitrary 2×2 matrices. Compute $A \otimes B$.